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EXISTENTIALLY COMPLETE NILPOTENT GROUPS[†]

BY

D. SARACINO

ABSTRACT

Let n be a positive integer, let K_n denote the theory of groups nilpotent of class at most n, and let K_n^+ denote the theory of torsion-free groups nilpotent of class at most n. We show that if $n \geqq 2$ then neither K_n nor K_n^+ has a model companion. For K_n we obtain the stronger result that the class of finitely generic models is disjoint from the class of infinitely generic models. We also give some other results about existentially complete nilpotent groups.

The study of model companions and existentially complete structures in group theory was initiated in [1], where Eklof and Sabbagh proved that the theory of abelian groups has a model companion but the theory of groups does not. Soon afterward, Macintyre [3] strengthened the negative result for groups by showing that the classes of existentially complete and infinitely generic groups are distinct ($E \neq G$). In [7] we considered the theory T_n of groups soluble of length $\leqq n$ (n fixed, $\geqq 2$), and showed that T_n has no model companion, for any n. Specializing to the case $n = 2$, we proved that for the theory of metabelian groups, there is an $\exists\forall\exists$ sentence which holds in every infinitely generic model and fails in every finitely generic one, and that consequently $E \neq G$ and the finite and infinite forcing companions are distinct.

In this paper, we consider the most interesting classes of groups left untreated by the above results.

THEOREM 1. *For any $n \geqq 2$ the theory of groups nilpotent of class $\leqq n$ has no model companion.*

A slight complication of the proof yields the following result, which is of independent interest because of the connection with nilpotent Lie algebras. (See [4], [8].)

[†] Dedicated to the Memory of Abraham Robinson.

THEOREM 2. *For any $n \geqq 2$ the theory of torsion-free groups nilpotent of class $\leqq n$ has no model companion.*

Finally, a different kind of argument, using finite forcing, proves

THEOREM 3. *Let K_n denote the theory of groups nilpotent of class $\leqq n$. Then for $n \geqq 2$ there is an $\exists\forall\exists$ sentence of first-order group theory which holds in every infinitely generic model of K_n and fails in every finitely generic one. Consequently, $E \neq G$ and the finite and infinite forcing companions of K_n are distinct.*

Of course Theorem 3 implies Theorem 1, but because our proof of Theorem 3 does not carry over to the torsion-free case we have chosen to present Theorem 1 separately and then indicate the changes in its proof required to prove Theorem 2.

1. Preliminaries

We assume that the reader is familiar with the basic machinery used in studying existentially complete structures, and in particular with the notions of finite and infinite forcing in model theory, finitely and infinitely generic structures, and finite and infinite forcing companions. The basic references are [5] and [6].

Let K be a first-order theory. If Σ denotes the class of substructures of models of K, i.e. the class Mod $(K_\forall)$ of models of the set of universal consequences of K, then it is well known that K has a model companion iff the class of existentially complete structures in Σ is elementary in the wider sense. Thus to show that a theory K has no model companion it is sufficient to show that the class of existentially complete structures in Σ is not closed under the formation of ultrapowers.

For definiteness we axiomatize the theory K of groups in a first-order language with equality which has a binary function symbol $\circ$ for group multiplication, a unary function symbol $^{-1}$ for inverses, and a constant symbol 1 for the identity element. It is easy to write down a set K of universal sentences in this language which axiomatizes the theory of groups.

To describe the theory K_n of groups nilpotent of class at most n we recall that if a and b are elements of a group G then their commutator $[a, b]$ is defined to be the element $a^{-1}b^{-1}ab$ of G. If A and B are subgroups of G then $[A, B]$ is the subgroup generated by all $[a, b]$ with $a \in A$, $b \in B$. The lower central series of G is the chain of normal subgroups $G = G^1$, $G^2 = [G, G], \cdots, G^n =$

$[G^{n-1}, G], \cdots, G^{\omega} = \bigcap_{n=1}^{\infty} G^n, \cdots, G^{\omega+1} = [G^{\omega}, G], \cdots$, etc. G is nilpotent of class $\leqq n$ if $G^{n+1} = \{1\}$.

If we define, by induction, an $(n+1)$-fold commutator $[x_1, \cdots, x_{n+1}]$ to be $[[x_1, \cdots, x_n], x_{n+1}]$, then it is not difficult to see that G is nilpotent of class $\leqq n$ iff every $(n+1)$-fold commutator in G is 1. Thus, if we let φ_n be the sentence abbreviated by $\forall x_1 \cdots \forall x_{n+1} [x_1, \cdots, x_{n+1}] = 1$, then $K_n = K \cup \{\varphi_n\}$ is a universal axiomatization of the class of groups nilpotent of class $\leqq n$.

We will make use of the following simple lemma on commutators.

LEMMA. *For any elements a, b, c of a group G,*

$$[ac, b] = c^{-1}[a, b]c[c, b].$$

Consequently if $[a, b]$ commutes with c then $[ac, b] = [a, b][c, b]$. In particular if a commutes with $[a, b]$ then $[a^r, b] = [a, b]^r$ for any positive integer r, and in fact this is also true if r is negative.

2. Nonexistence of model companions

To prove Theorem 1 we will show that the class of existentially complete (e. c.) models of the universal theory K_n is not closed under the formation of ultrapowers if $n \geqq 2$. The following propositions contain the necessary information about e.c. models of K_n. As a matter of notation, if G is a group and $a \in G$ then $Z(G)$ denotes the center of G and $Z(a)$ denotes the centralizer of a. We fix $n \geqq 2$.

PROPOSITION 1. *Let G be an e.c. model of K_n. Then for any positive integer k there exists in G an $(n-1)$-fold commutator which has order $\geqq k$ modulo $Z(G)$.*

PROOF. Let H be a torsion-free nilpotent group of class n, and let b be an $(n-1)$-fold commutator which is not in $Z(H)$. Observe that b has infinite order modulo $Z(H)$, because if $b^m \in Z(H)$ and we choose d such that $bd \neq db$ then $[b, d] \neq 1$ is in $Z(H)$ (because H is of class n) and consequently by the Lemma $[b, d]^m = [b^m, d] = 1$, so there is torsion in H, a contradiction.

Choosing elements $d_1, \cdots, d_k$ which don't commute with $b, b^2, \cdots b^k$ respectively, we see that if φ is the sentence $\exists x \exists w_1 \ldots \exists w_{n-1} \exists y_1 \ldots \exists y_k\, (x = [w_1, \cdots, w_{n-1}] \wedge \bigwedge_{j=1}^{k} y_j x^j \neq x^j y_j)$, then $H \models \varphi$, so $G \oplus H \models \varphi$, so $G \models \varphi$ since G is e.c.

PROPOSITION 2. *Let G be an e.c. model of K_n and let G^* denote an ultrapower of G formed with respect to a nonprincipal ultrafilter on the set I of positive integers.*

Then there exist elements b, c in G^* such that i) b, $c \notin Z(G^*)$, ii) *b is an $(n-1)$-fold commutator,* iii) *c is not in the subgroup of G^* generated by $Z(G^*) \cup \{b\}$, and* iv) *$Z(b) \subseteq Z(c)$ in G^*.*

PROOF. For $i \in I$ let b_i be an $(n-1)$-fold commutator in G which is of order $\geqq 2i$ modulo $Z(G)$. Let $c_i = b_i^i$, and let b and c be the elements of G^* represented by the sequences $\{b_i\}$ and $\{c_i\}$ respectively. Then it is easy to see that (i), (ii) and (iv) are satisfied. For (iii), suppose $c = b^m t$, where m is an integer and $t \in Z(G^*)$. Then $b^{-m}c = t \in Z(G^*)$, but for $i > |m|$, $b_i^{-m}c_i = b_i^{i-m} \notin Z(G)$ since $1 < i - m < 2i$. This situation is impossible by Łos' Theorem, and the proof is complete.

PROPOSITION 3. *Let G be a model of K_n and b, c elements of G such that (i), (ii), and (iii) of Proposition 2 hold with G^* replaced by G. Then there exists a model H of K_n extending G and $x \in H$ which commutes with b but not with c.*

PROOF. Let (z) denote the infinite cycle on z, let $F = G * (z)$, and let $H_1 = F/F^{n+1}$. If $d \in G$ is in F^{n+1} then by applying the homomorphism $F \to G$ obtained by mapping $G \to G$ identically and $z \mapsto 1$, we get $d \in G^{n+1}$, so $d = 1$ since G is nilpotent of class at most n. Thus we can consider $G \subseteq H_1$, and clearly H_1 is a model of K_n. Observe for future reference that any homomorphism $F \to G$ induces a homomorphism $H_1 \to G$.

If $\bar{z}$ denotes the canonical image of z in H_1 and H_2 denotes the subgroup of H_1 generated by $[b, \bar{z}]$, then H_2 is a normal subgroup of H_1 since $[b, \bar{z}] \in Z(H_1)$. Let H denote the factor group H_1/H_2. $H \supseteq G$, because $H_2 \cap G = \{1\}$, as we see by considering the homomorphism of the last paragraph. Clearly H is a model of K_n, and in H the image $\bar{\bar{z}}$ of $\bar{z}$ mod H_2 commutes with b. But suppose $[c, \bar{\bar{z}}] = 1$. Then in H_1, $[c, \bar{z}] = [b, \bar{z}]^n$ for some integer n, so by the Lemma $[c, \bar{z}] = [b^n, \bar{z}]$, so $\bar{z}$ commutes with cb^{-n}. But $cb^{-n} \notin Z(G)$, so cb^{-n} does *not* commute with $\bar{z}$ (consider a homomorphism $H_1 \to G$ which is the identity on G and takes $\bar{z}$ to an element which doesn't commute with cb^{-n}). This contradiction shows that we can take $x = \bar{\bar{z}}$, and the proposition is proved.

The proof of Theorem 1 is now immediate because, in the notation of Proposition 2, the ultrapower G^* fails to satisfy $\exists x(xb = bx \wedge xc \neq cx)$, while Proposition 3 shows that some extension of G^* does satisfy this sentence. Hence G^* is not e.c.

The proof of Theorem 2 follows the same sequence of ideas as the proof of Theorem 1, but requires some modifications in Propositions 2 and 3. Let K_n^+ denote the theory of torsion-free groups nilpotent of class at most n.

PROPOSITION 4. *Proposition* 1 *remains valid if* K_n *is replaced by* K_n^+.

PROPOSITION 5. *Proposition* 2 *remains valid if we replace* K_n *by* K_n^+ *and strengthen* (*iii*) *to read* (*iii*)′: *the subgroup generated by c intersects the subgroup of* G^* *generated by* $Z(G)^* \cup \{b\}$ *trivially.*

PROOF. For $i \in I$ let b_i be an $(n-1)$-fold commutator in G which is of order $\geqq i(i+1)$ modulo $Z(G)$. Let $c_i = b_i^i$, and again let b and c denote the elements of G^* represented by the sequences $\{b_i\}$ and $\{c_i\}$. Clauses (i), (ii), and (iv) are easy as before, and to prove (iii)′ observe that for integers k, m we have $c_i^k b_i^{-m} = b_i^{ik-m}$ for any i; assuming not both k and m are 0 and that $i > |k|, |m|$, we have $1 \leqq |ik - m| < i^2 + i$, so $c_i^k b_i^{-m} \notin Z(G)$. Thus $c^k b^{-m} \notin Z(G^*)$ by Łos' Theorem.

PROPOSITION 6. *Let G be a model of* K_n^+ *and b, c elements of G such that* (*i*), (*ii*), *and* (*iii*)′ *of Proposition* 5 *hold with* G^* *replaced by G. Assume that* $[c, a] \in Z(G)$ *for all* $a \in G$. *Then there exist a model M of* K_n^+ *extending G and* $x \in M$ *which commutes with b but not with c.*

PROOF. Form H_1 and $H \supset G$ as in the proof of Proposition 3 and let M denote the quotient of H by its torsion subgroup. (The set of torsion elements is a normal subgroup in any nilpotent group.) Then we can consider $G \subseteq M$ because G is torsion-free. Let x denote the image in the quotient of the element $\bar{\bar{z}}$ of Proposition 3. All we have to prove is that $[c, x] \neq 1$ in M. If $[c, x] = 1$, then $[c, \bar{\bar{z}}]^m = 1$ in H for some positive integer m. Thus $[c, \bar{z}]^m = [b, \bar{z}]^n$ in H_1 for some integer n. Consider a homomorphism $H_1 \to G$ given by $G \to G$ identically, $z \mapsto$ an element a which does not commute with $c^m b^{-n}$ (possible since $c^m b^{-n} \notin Z(G)$). Then $[c, a]^m = [b, a]^n$ in G, so since by assumption both $[c, a]$ and $[b, a]$ are in $Z(G)$, the Lemma yields $[c^m, a] = [b^n, a]$, whence $c^{-m} a^{-1} c^m = b^{-n} a^{-1} b^n$ and $c^m b^{-n}$ commutes with a, a contradiction.

Now to finish the proof of Theorem 2, let G^*, b, c be as in Proposition 5. Observe that the extra hypothesis imposed on c by Proposition 6 is satisfied in G^*, i.e. $[c, a] \in Z(G^*)$ for all $a \in G^*$. Therefore, there exists a model M of K_n^+ extending G^* and satisfying $\exists x\,(xb = bx \wedge xc \neq cx)$, so as in the proof of Theorem 1, G^* is not e.c.

3. $E \neq G$

In this section we give the proof of Theorem 3. K_n is as in the previous section.

PROPOSITION 7. *Let G be a model of* K_n, $n \geqq 2$, *let c be an n-fold commutator in*

G, and let be an $(n-1)$-fold commutator which is of infinite order modulo $Z(G)$. Then there exist a model H of K_n extending G and $h \in H$ such that $c = [b, h]$.

PROOF. Consider $G \subseteq G*(z)/(G*(z))^{n+1} = H_1$, as before. We claim that

$$(c^{-1}[b, \bar{z}]) \cap G = \{1\},$$

where $(c^{-1}[b, \bar{z}])$ is the cycle on $c^{-1}[b, \bar{z}]$. For, since c is in $Z(H_1)$, $(c^{-1}[b, \bar{z}]) = \{c^{-n}[b, \bar{z}]^n : n \in \mathbf{Z}\}$. If $c^{-n}[b, \bar{z}]^n = d \in G$, $n \neq 0$, then $c^{-n} = d$ (again consider the map $G \to G$ identically, $x \mapsto 1$), so $[b, \bar{z}]^n = 1$, i.e. $[b^n, \bar{z}] = 1$. Thus $b^n \in Z(G)$, contradicting the fact that b has infinite order modulo $Z(G)$.

Observe that $(c^{-1}[b, \bar{z}])$ is normal in H_1 since it is central. Now if $H = H_1/(c^{-1}[b, \bar{z}])$, then by the claim we can consider $G \subseteq H$, and if h denotes the canonical image of $\bar{z}$ in H then clearly $c = [b, h]$.

REMARK. Proposition 7 holds for K_n^+ as well as for K_n.

PROPOSITION 8. *Let G be an e.c. model of K_n, $n \geqq 2$. Let b be an $(n-1)$-fold commutator in G. Then b has infinite order modulo $Z(G)$ iff*

$$G \models \forall x_1 \cdots \forall x_n \exists y([x_1, \cdots, x_n] = [b, y]).$$

PROOF. "Only if" follows from the previous proposition, since G is e.c. For "if", note that for any $m > 0$ there are $g_1, \cdots, g_n \in G$ such that $[g_1, \cdots, g_n]^m \neq 1$. Now if $b^m \in Z(G)$ then for all $y \in G$, $[b, y]^m = [b^m, y] = 1$; so if

$$G \models \forall x_1 \cdots \forall x_n \exists y\, [x_1, \cdots, x_n] = [b, y]),$$

then $[g_1, \cdots, g_n]^m = 1$ for all $g_1, \cdots, g_n \in G$, a contradiction.

Now for $n \geqq 2$ let Ψ_n be the sentence

$$\exists w_1 \cdots \exists w_{n-1} \forall x_1 \cdots \forall x_n \exists y([x_1, \cdots, x_n] = [w_1, \cdots, w_{n-1}, y]).$$

PROPOSITION 9. *If G is an infinitely generic model of K_n then $G \models \Psi_n$.*

PROOF. Let G_0 be a model of K_n which contains an $(n-1)$-fold commutator which is of infinite order modulo $Z(G_0)$. Let G_1 be an e.c. model of K_n such that $G_0 \subseteq G_1$. Then in G_1, c is of infinite order modulo $Z(G_1)$, so by Proposition 8, $G_1 \models \Psi_n$. If in particular we take G_1 infinitely generic, then we see that there exists an infinitely generic model of K_n which satisfies Ψ_n. Since all infinitely generic models of K_n are elementarily equivalent because K_n has the joint embedding property, this finishes the proof.

PROPOSITION 10. *If G is a finitely generic model of K_n then $G \models \neg \Psi_n$.*

PROOF. It suffices to show that no condition can force Ψ_n. So suppose $p \vdash \Psi_n$; then for some $c_1, \cdots, c_{n-1}$ in a set C of forcing constants,

$$p \vdash \forall x_1 \cdots \forall x_n \exists y\ ([x_1, \cdots, x_n] = [c_1, \cdots, c_{n-1}, y])$$

and we can assume that $c_1, \cdots, c_{n-1}$ occur in p. Thus for any $p_1 \supset p$, and any $d_1, \cdots, d_n$ in C, there are e in C and an extension p_2 of p_1 such that

$$p_2 \vdash [d_1, \cdots, d_n] = [c_1, \cdots, c_{n-1}, e].$$

Now since finitely generated nilpotent groups are residually finite [2], there exists an integer $m > 0$ such that $p_0 = p \cup \{[c_1, \cdots, c_{n-1}]^m = 1\}$ is a condition. (Compare proposition 4 of [7].) If we choose $d_1, \cdots, d_n$ not mentioned in p then $p_1 = p_0 \cup \{[d_1, \cdots, d_n]^m \neq 1\}$ is a condition. Taking p_2 and e as above,

$$p_2 \vdash [d_1, \cdots, d_n] = [c_1, \cdots, c_{n-1}, e],$$

so the formula $[d_1, \cdots, d_n] = [c_1, \cdots, c_{n-1}, e]$ is in p_2, as are $[c_1, \cdots, c_{n-1}]^m = 1$ and $[d_1, \cdots, d_n]^m \neq 1$. But this is impossible, because these three formulas are inconsistent: the first two give $[d_1, \cdots, d_n]^m = [c_1, \cdots, c_{n-1}, e]^m = [[c_1, \cdots, c_{n-1}]^m, e] = 1$, contradicting the third.

Propositions 9 and 10 yield Theorem 3. Together with Proposition 8, they also show that an infinitely generic model of K_n can be distinguished from a finitely generic one by the fact that the quotient of the former by its center must contain an element $[x_1, \cdots, x_{n-1}]$ of infinite order, while the quotient of the latter by its center can never contain such an element. In particular for $n = 2$, one quotient can never be periodic and the other must always be.

In the case of K_n^+, every noncentral $(n-1)$-fold commutator has infinite order modulo the center, since otherwise there would be torsion in the center. Hence every e.c. model of K_n^+ satisfies

$$\forall w_1 \cdots \forall w_{n-1}\{\exists z\,([w_1, \cdots, w_{n-1}, z] \neq 1) \rightarrow$$
$$\forall x_1 \cdots \forall x_n \exists y([x_1, \cdots, x_n] = [w_1, \cdots, w_{n-1}, y])\},$$

and in particular every such model satisfies Ψ_n.

We conclude with a result on the centers of e.c. nilpotent groups. In [7] we proved that every element of the commutator subgroup of an e.c. metabelian (solvable of rank $\leqq 2$) group is itself actually a commutator. We have the following analogue for K_n:

PROPOSITION 11. *Let G be an e.c. model of K_n. Then i) every element of the center of G is an n-fold commutator, and consequently ii) $Z(G) = G^n$.*

PROOF. Let M be a model of K_n containing an $(n-1)$-fold commutator b which is of infinite order modulo the center, and let $N = G \oplus M$. Then b has infinite order modulo $Z(N)$. If $c \in Z(G)$ then c is central in any model of K_n extending G, because G is e.c. This is all we need to repeat the proof of Proposition 7 and find an extension H of N and $h \in H$ such that $c = [b, h]$. Thus $H \models \exists x_1 \cdots \exists x_{n-1} \exists y \, (c = [x_1, \cdots, x_{n-1}, y])$, so this sentence holds in G.

PROPOSITION 12. *Proposition* 11 *holds with* K_n *replaced by* K_n^+.

PROOF. If G is torsion-free then we can, if necessary, factor the extension H of Proposition 11 by its torsion subgroup to obtain a torsion-free extension of G containing the desired elements b and h.

The following corollary of Proposition 11 was found during a conversation with Greg Cherlin.

PROPOSITION 13. *i) For any* $n \geqq 2$, *the center of any finitely generic model of* K_n *is periodic.*

ii) Every finitely generic model of K_2 *is periodic.*

PROOF. i) Let $c \in Z(G)$ where G is a finitely generic model of K_n. By Proposition 11, we have $c = [[x_1, \cdots, x_{n-1}], x_n]$ for some $x_1, \cdots, x_n$ in G. Propositions 8 and 10 give us an integer m such that $[x_1, \cdots, x_{n-1}]^m \in Z(G)$. Then $c^m = 1$.

ii) We have already remarked that for the case $n = 2$, $G/Z(G)$ is periodic; thus by (i), G is itself periodic.

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COLGATE UNIVERSITY
UNIVERSITÄT HEIDELBERG